MATHEMATICAL MODEL DESCRIBING THE PROCESS OF INTERACTION BETWEEN GEORGIAN, LAZ, MINGRELIAN, AND SVAN POPULATIONS

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Abstract. This paper examines the fourth period of Kartvelian population transformation, during which Laz and Mingrelian languages emerged from the Zan linguistic base, leading to the formation of four Kartvelian languages: Georgian, Laz, Mingrelian, and Svan. This process involves internal linguistic-cultural interactions and external influences (assimilation, dissimilation).

A mathematical model, formulated as a four-dimensional nonlinear dynamic system, describes the interactions and demographic changes of Kartvelian populations, incorporating demographic parameters, linguistic-cultural interactions, and external factors (wars, migrations, epidemics).

The model enables analysis of historical development. Methods for finding first integrals in four- and three-dimensional systems are discussed. Using two first integrals, the system is reduced to a two-dimensional system, capturing Georgian-Laz linguistic dynamics, with Mingrelian and Svan population sizes derived algebraically.

Theorems establish the coexistence dynamics of Georgian, Laz, Mingrelian, and Svan populations. The Poincaré-Bendixson theorem confirms closed integral trajectories, ensuring no Kartvelian language group becomes extinct.

Keywords: *Mathematical model, transformation of the Proto-Kartvelian population, Georgian, Laz, Mingrelian, and Svan populations and languages, mathematical models, Lotka-Volterra system, analytical solution, first integrals, four-dimensional hyperspace, Poincaré–Bendixson theorem.*

Introduction. The study of a range of social processes - such as information warfare, assimilation of peoples, globalization, predicting political election results, resolution of political conflicts through economic cooperation, secession of regions, territorial integrity of states, etc. - is of great theoretical and practical interest. From our point of view, the only scientific approach for adequately and quantitatively describing these pressing issues is the creation of **mathematical models** and the study of the corresponding **mathematical problems** [1–8].

Synergetics gave a powerful push using of mathematical models in social sciences. Mathematical modeling of social processes compared to modeling in natural science is more original due to the complexity of model justifications.

From a historical point of view, we see mathematical modeling as an innovative approach to describe the area of distribution of the Proto-Kartvelian speaking population and the process of further transformation of the language, determining the number of the population speaking the corresponding language in each time period.

To describe the process of transformation of the Proto-Kartvelian population into four populations speaking four different languages, Georgian, Mengrelian, Laz and Svan, we divided mathematical modeling into four separate stages: the first stage (L–XXV BC), at the end of which there is a division of the Proto-Kartvelian population into three parts: the first part emigrated to Europe and gradually completely or partially assimilated, the second part speaking the Svan language, the third part speaking the Colchian-Georgian language; second stage (XXV–X century BC), at the end of which the Colchian-Georgian population fell into two parts: the Colchian and Georgian populations; the third stage (X–I century BC. e.), at the end of which the Colchian population broke up into Mengrelian and Laz; moreover, in (X–III century BC) the Colchian population prevailed over others, and in (III–I century BC. e.) the Georgian population prevailed over others; the fourth stage (1st century BC – to this day), when four multilingual populations Georgian, Mengrelian, Laz, Svan live peacefully in the Caucasus and western Asia in a small area.

Mathematical modeling of the first stage is considered in [9]. Mathematical and computer modeling of the second stage is considered in [10–12]. Mathematical modeling of the third stage is considered in [13]. Computer modeling of the third stage is considered in [14].

1. MATHEMATICAL MODEL OF THE POPULATION DYNAMICS OF GEORGIAN, LAZ, MINGRELIAN, AND SVAN POPULATIONS (1ST CENTURY BC TO PRESENT)

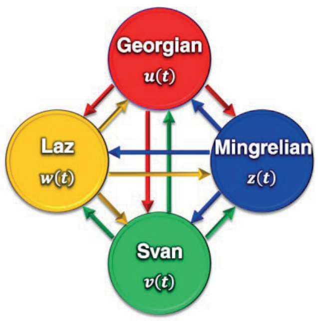

Fig. 1.1

Problem Statement. This study presents a general mathematical model designed to describe the quantitative changes in the populations of Georgian, Laz, Mingrelian, and Svan communities, including the processes of ongoing linguistic-cultural influences and interactions, spanning from the 1st century BC to the present day (see Fig. 1.1).

The aforementioned model is presented below in the form of a four-dimensional nonlinear dynamic system with variable coefficients, where each equation correspondingly describes the processes of quantitative changes in the populations of Georgian, Laz, Mingrelian, and Svan populations. This system is formalized as follows:

$$\begin{cases} \frac{du(t)}{dt} = \alpha_{10}(t)u(t) + \gamma_{15}(t)w(t)u(t) + \gamma_{16}(t)z(t)u(t) + \gamma_{17}(t)v(t)u(t) - \delta_6(t)u(t) \\ \frac{dw(t)}{dt} = \alpha_{11}(t)w(t) - \gamma_{18}(t)u(t)w(t) + \gamma_{19}(t)z(t)w(t) + \gamma_{20}(t)v(t)w(t) - \delta_7(t)w(t) \\ \frac{dz(t)}{dt} = \alpha_{12}(t)z(t) - \gamma_{21}(t)u(t)z(t) - \gamma_{22}(t)w(t)z(t) + \gamma_{23}(t)v(t)z(t) - \delta_8(t)z(t) \\ \frac{dv(t)}{dt} = \alpha_{13}(t)v(t) - \gamma_{24}(t)u(t)v(t) - \gamma_{25}(t)w(t)v(t) - \gamma_{26}(t)z(t)v(t) - \delta_9(t)v(t) \\ u(t_4) = u_4,\ \ w(t_4) = w_4,\ \ z(t_4) = z_4,\ \ v(t_4) = v_4 \end{cases}, \tag{1.1}$$

where $u(t), w(t), z(t), v(t) \in C^1[t_4, t_5]$; $\alpha_i(t) \in C[t_4, t_5]$, $i = \overline{10-13}$, $\alpha_{13}(t) > 0$; $\gamma_j(t) \in C[t_4, t_5]$, $\gamma_j(t) > 0$, $j = \overline{15-26}$; $\delta_k(t) \in C[t_4, t_5]$, $\delta_k(t) > 0$, $k = \overline{6-9}$; $t \in [t_4, t_5]$, $t_4 =$ 4900 years (1st century BC) and $t_5 = 7025$ years (2025 CE); $u(t)$ represents the number of the Georgian-speaking population at time t; $w(t)$ represents the number of the Laz-speaking population at time t; $z(t)$ represents the number of the Mingrelian-speaking population at time t; $v(t)$ represents the number of the Svan-speaking population at time t; $\alpha_i(t)$, $i = \overline{10-13}$, corresponds to the demographic factors of the Georgian, Laz, Mingrelian, and Svan populations, respectively, i.e., the natural birth-death rate; $\gamma_j(t)$, $j = \overline{15-26}$, corresponds to the coefficients of interaction (assimilation) among these linguistic groups; $\delta_k(t) > 0$, $k = \overline{6-9}$, corresponds to the losses of the aforementioned linguistic groups (due to wars, ecosystems, migrations, and other external factors).

The mathematical model presented in (1.1) belongs to the class of nonlinear dynamic systems, which, through the analysis of historical-archaeological, linguistic, and ethnographic materials, it accounts for demographic factors (birth rates, mortality rates), migration, socio-economic conditions (urbanization, living conditions, economic stability), and historical events (wars, epidemics, cultural assimilations).

Thus, this mathematical model enables the study of the historical development dynamics of populations speaking Kartvelian languages, as well as the impact of key factors that determine the processes of integration or differentiation among them.

2. FIRST INTEGRALS OF FOUR- AND THREE-DIMENSIONAL NONLINEAR DYNAMIC SYSTEMS

Suppose that the coefficients of the dynamic system (1.1) are constant. In this case, the Cauchy problem for system (1.1) can be rewritten as follows:

$$\begin{cases} \frac{du(t)}{dt} = (\alpha_{10} - \delta_6)u(t) + \gamma_{15}w(t)u(t) + \gamma_{16}z(t)u(t) + \gamma_{17}v(t)u(t) \\ \frac{dw(t)}{dt} = (\alpha_{11} - \delta_7)w(t) - \gamma_{18}u(t)w(t) + \gamma_{19}z(t)w(t) + \gamma_{20}v(t)w(t) \\ \frac{dz(t)}{dt} = (\alpha_{12} - \delta_8)z(t) - \gamma_{21}u(t)z(t) - \gamma_{22}w(t)z(t) + \gamma_{23}v(t)z(t) \\ \frac{dv(t)}{dt} = (\alpha_{13} - \delta_9)v(t) - \gamma_{24}u(t)v(t) - \gamma_{25}w(t)v(t) - \gamma_{26}z(t)v(t) \\ u(t_4) = u_4,\ w(t_4) = w_4,\ z(t_4) = z_4,\ v(t_4) = v_4 \end{cases} \cdot \tag{2.1}$$

As a result of transforming the dynamic system (2.1), we obtain:

$$\begin{cases} \frac{du(t)}{u\,dt} = (\alpha_{10} - \delta_6) + \gamma_{15}w(t) + \gamma_{16}z(t) + \gamma_{17}v(t) \\ \frac{dw(t)}{w\,dt} = (\alpha_{11} - \delta_7) - \gamma_{18}u(t) + \gamma_{19}z(t) + \gamma_{20}v(t) \\ \frac{dz(t)}{z\,dt} = (\alpha_{12} - \delta_8) - \gamma_{21}u(t) - \gamma_{22}w(t) + \gamma_{23}v(t) \\ \frac{dv(t)}{v\,dt} = (\alpha_{13} - \delta_9) - \gamma_{24}u(t) - \gamma_{25}w(t) - \gamma_{26}z(t) \\ (t_4) = u_4,\ w(t_4) = w_4,\ z(t_4) = z_4,\ v(t_4) = v_4 \end{cases} \cdot \tag{2.2}$$

To ensure the adequacy and non-triviality of the mathematical model (2.1), it is necessary that the constant coefficients of the system satisfy the following additional condition:

$$\begin{cases} \alpha_{10} - \delta_6 < 0 \\ \alpha_{13} - \delta_9 > 0 \end{cases} \cdot \tag{2.3}$$

The first condition of system (2.3) indicates that the number of the Georgian-speaking population decreases at time t due to demographic and other factors. However, this decline is balanced by the assimilation of the Laz, Mingrelian, and Svan linguistic groups, which contributes to an increase in the Georgian population. Simultaneously, according to the second condition of system (2.3), the Svan-speaking population naturally increases, but its growth is slowed by the influence of the Georgian, Laz, and Mingrelian populations, which is associated with linguistic, cultural, and socio-economic transformation processes.

To investigate the system of equations (2.1), it is necessary to find the first integral. In this regard, if we add the first and fourth equations of system (2.2) and subtract the second and third equations of the same system from the resulting sum, we obtain the following equality:

$$\frac{du(t)}{u\,dt}+\frac{dv(t)}{v\,dt}-\frac{dw(t)}{w\,dt}-\frac{dz(t)}{z\,dt}=(\alpha_{10}-\delta_6)+(\alpha_{13}-\delta_9)-(\alpha_{11}-\delta_7)-$$
$$-(\alpha_{12}-\delta_8)+(\gamma_{18}+\gamma_{21}--\gamma_{24})u(t)+(\gamma_{15}+\gamma_{22}-\gamma_{25})w(t)+$$
$$+(\gamma_{16}-\gamma_{19}-\gamma_{26})z(t)+(\gamma_{17}-\gamma_{20}-\gamma_{23})v(t). \tag{2.4}$$

Let us assume that the constant coefficients of equation (2.4) satisfy the following additional system:

$$\begin{cases}(\alpha_{10}-\delta_6)+(\alpha_{13}-\delta_9)-(\alpha_{11}-\delta_7)-(\alpha_{12}-\delta_8)=0\\ \gamma_{18}+\gamma_{21}-\gamma_{24}=0\\ \gamma_{15}+\gamma_{22}-\gamma_{25}=0\\ \gamma_{16}-\gamma_{19}-\gamma_{26}=0\\ \gamma_{17}-\gamma_{20}-\gamma_{23}=0\end{cases}. \tag{2.5}$$

These conditions do not contradict the system (2.3).

Taking into account the conditions of (2.3) and (2.5), equation (2.4) can be rewritten as follows:

$$\frac{du(t)}{u\,dt}+\frac{dv(t)}{v\,dt}-\frac{dw(t)}{w\,dt}-\frac{dz(t)}{z\,dt}=0. \tag{2.6}$$

From this, in the phase hyperspace of solutions $\big(O,v(t),z(t),w(t),u(t)\big)$, we obtain the first integral of the Cauchy problem for the dynamic system (2.1):

$$\left[ln\frac{u(t)v(t)}{w(t)z(t)}\right]'=0\Leftrightarrow\frac{u(t)v(t)}{w(t)z(t)}=const\Leftrightarrow u(t)v(t)=pw(t)z(t), \tag{2.7}$$

where $p\equiv\frac{u_4v_4}{w_4z_4}=const>0$.

Equation (2.7) represents a nonlinear relationship (equation) among the four unknown functions, indicating that this equality remains invariant over time t, and the unknowns are interdependent. Equation (2.7) defines a certain hypersurface in the phase space of solutions $\big(O,v(t),z(t),w(t),u(t)\big)$ for the four-dimensional dynamic system (2.1). As previously noted, (2.7) characterizes a four-dimensional hyperspace that expresses the mutual dependence of the system's state variables in (2.1), where the variation of one unknown significantly affects the others. The stability of such interdependence is crucial for understanding the long-term behavior of the system, as it underlies the balance and stability of the processes involved.

Let us consider the first three equations of system (2.2), and multiply the second equation by 2. As a result, we obtain the following system of equations:

$$\begin{cases} \frac{du(t)}{u\,dt} = (\alpha_{10} - \delta_6) + \gamma_{15} w(t) + \gamma_{16} z(t) + \gamma_{17} v(t) \\ 2\frac{dw(t)}{w\,dt} = 2(\alpha_{11} - \delta_7) - 2\gamma_{18} u(t) + 2\gamma_{19} z(t) + 2\gamma_{20} v(t). \\ \frac{dz(t)}{z\,dt} = (\alpha_{12} - \delta_8) - \gamma_{21} u(t) - \gamma_{22} w(t) + \gamma_{23} v(t) \end{cases} \tag{2.8}$$

If we add the first and third equations of system (2.8) and then subtract the second equation of the same system from the resulting sum, we obtain the following equality:

$$\frac{du(t)}{u\,dt} + \frac{dz(t)}{z\,dt} - 2\frac{dw(t)}{w\,dt} = (\alpha_{10} - \delta_6) + (\alpha_{12} - \delta_8) - 2(\alpha_{11} - \delta_7) + (2\gamma_{18} - \gamma_{21})u(t) +$$
$$+(\gamma_{15} - \gamma_{22})w(t) + (\gamma_{16} - 2\gamma_{19})z(t) + (\gamma_{17} + \gamma_{23} - 2\gamma_{20})v(t). \tag{2.9}$$

To find the second first integral, it is necessary to examine the conditions under which the right-hand side of equation (2.9) becomes zero. If we assume that the right-hand side of identity (2.9) vanishes for all values of t, then each term must independently be equal to zero.

Thus, the constant coefficients in equation (2.9) must satisfy the following additional conditions:

$$\begin{cases} (\alpha_{10} - \delta_6) + (\alpha_{12} - \delta_8) - 2(\alpha_{11} - \delta_7) = 0 \\ 2\gamma_{18} - \gamma_{21} = 0 \\ \gamma_{15} - \gamma_{22} = 0 \\ \gamma_{16} - 2\gamma_{19} = 0 \\ \gamma_{17} + \gamma_{23} - 2\gamma_{20} = 0 \end{cases}. \tag{2.10}$$

which do not contradict systems (2.3) and (2.5).

If system (2.10) holds, then the identity (2.9) takes the following form:

$$\frac{du(t)}{u\,dt} + \frac{dz(t)}{z\,dt} - 2\frac{dw(t)}{w\,dt} = 0 \Leftrightarrow \left[ln\frac{u(t)z(t)}{w^2(t)}\right]' = 0,$$

from which, in the phase space of solutions $(O, z(t), w(t), u(t))$, we obtain the second first integral of the Cauchy problem for the dynamic system (2.1):

$$u(t)z(t) = qw^2(t). \tag{2.11}$$

where $q \equiv \frac{u_4 z_4}{w_4^2} = const > 0$.

From (2.7) and (2.11), we derive the following system:

$$\begin{cases} u(t)v(t) = pw(t)z(t) \\ u(t)z(t) = qw^2(t) \end{cases} \Leftrightarrow \begin{cases} v(t) = p\frac{w(t)z(t)}{u(t)} \\ z(t) = q\frac{w^2(t)}{u(t)} \end{cases} \Leftrightarrow \begin{cases} v(t) = pq\frac{w^3(t)}{u^2(t)} \\ z(t) = q\frac{w^2(t)}{u(t)} \end{cases}. \tag{2.12}$$

Accounting for the system of relations (2.12), the four-dimensional dynamic system (2.1) is reduced to the following two-dimensional dynamic system containing only the unknown functions $u(t)$ and $w(t)$:

$$\begin{cases} \frac{du(t)}{u\,dt} = (\alpha_{10} - \delta_6)u(t) + \gamma_{15}w(t)u(t) + \gamma_{16}qw^2(t) + \gamma_{17}pq\frac{w^3(t)}{u(t)} \\ \frac{dw(t)}{w\,dt} = (\alpha_{11} - \delta_7)w(t) - \gamma_{18}u(t)w(t) + \gamma_{19}q\frac{w^3(t)}{u(t)} + \gamma_{20}pq\frac{w^4(t)}{u^2(t)}, \\ u(t_4) = u_4, \;\; w(t_4) = w_4 \end{cases} \tag{2.13}$$

Let us now introduce the following notations:

$$\begin{cases} F_1\big(u(t), w(t)\big) \equiv (\alpha_{10} - \delta_6)u(t) + \gamma_{15}w(t)u(t) + \gamma_{16}qw^2(t) + \gamma_{17}pq\frac{w^3(t)}{u(t)} \\ F_2\big(u(t), w(t)\big) \equiv (\alpha_{11} - \delta_7)w(t) - \gamma_{18}u(t)w(t) + \gamma_{19}q\frac{w^3(t)}{u(t)} + \gamma_{20}pq\frac{w^4(t)}{u^2(t)} \end{cases}. \tag{2.14}$$

To study the nonlinear system of differential equations (2.13), let us consider the divergence of the vector field $\vec{F}(F_1, F_2)$:

$$G(w,u) \equiv div\vec{F} = \frac{\partial F_1(u,w)}{\partial u} + \frac{\partial F_2(u,w)}{\partial w} = (\alpha_{10} - \delta_6) + (\alpha_{11} - \delta_7) + \gamma_{15}w(t) -$$

$$-\gamma_{18}u(t) + 3\gamma_{19}q\frac{w^2(t)}{u(t)} + (4\gamma_{20} - \gamma_{17})pq\frac{w^3(t)}{u^2(t)}. \tag{2.15}$$

Let us now consider the curve in the phase plane $\big(O, w(t), u(t)\big)$, on which the divergence of the vector field vanishes:

$$G(w,u) = 0. \tag{2.16}$$

3. THEOREMS ON THE COEXISTENCE OF GEORGIAN, LAZ, MINGRELIAN, AND SVAN POPULATIONS

For equation (2.16), we consider three cases below [7].

First Case. Suppose the following condition is satisfied:

$$\begin{cases} (\alpha_{10} - \delta_6) + (\alpha_{11} - \delta_7) = 0 \\ 4\gamma_{20} - \gamma_{17} = 0 \end{cases}. \tag{3.1}$$

which does not contradict the previously established conditions (2.3), (2.5), and (2.10). In this case, the divergence of the vector field in (2.15) and (2.16) becomes zero on the following curve in the phase plane $\big(O, w(t), u(t)\big)$:

$$G(w,u) = \gamma_{15}w(t) - \gamma_{18}u(t) + 3\gamma_{19}q\frac{w^2(t)}{u(t)} = 0 \Leftrightarrow$$

$$\gamma_{18}u^2(t) - \gamma_{15}w(t)u(t) - 3\gamma_{19}qw^2(t) = 0. \tag{3.2}$$

Let us introduce a new notation in (3.2):

$$\varphi \equiv \frac{u}{w}.$$

such that $u = \varphi w$. Substituting into equation (3.2), we obtain the quadratic equation:

$$w^2(t)\,[\gamma_{18}\varphi^2(t) - \gamma_{15}\varphi(t) - 3\gamma_{19}q] = 0 \Leftrightarrow$$

$$\gamma_{18}u^2(t) - \gamma_{15}w(t)u(t) - 3\gamma_{19}qw^2(t) = 0. \tag{3.3}$$

since $w \neq 0$.

We seek the positive solution of (3.3):

$$\varphi = \frac{\gamma_{15} + \sqrt{\gamma_{15}^2 + 12\gamma_{18}\gamma_{19}q}}{2\gamma_{18}},$$

since only the first quadrant of the phase plane $(O, w(t), u(t))$ has physical meaning.

Thus, (3.2) represents a half-line located in the phase plane of solutions $(O, w(t), u(t))$ of system (2.13), passing through the origin of the coordinate system. The half-line is expressed as follows:

$$\begin{cases} u = \varphi w \\ \varphi = \frac{\gamma_{15} + \sqrt{\gamma_{15}^2 + 12\gamma_{18}\gamma_{19}q}}{2\gamma_{18}} = const > 0. \\ w(t) > 0 \end{cases} \tag{3.4}$$

The following theorem holds true.

Theorem 3.1. The problem (2.13), (2.14), (2.3), (2.5), (2.10), (3.1) has a solution in the form of a closed integral trajectory within some simply connected domain $D \subset (O, w(t), u(t))$ in the first quadrant of the phase plane of solutions $(O, w(t), u(t))$ with physical significance, where the solution is entirely contained.

Proof. Consider a curve located on the phase plane $(O, w(t), u(t))$, where the divergence of the vector field $\vec{F}(F_1, F_2)$ is equal to zero. Taking into account equations (2.15) and (3.1), the curve (3.2) will be a half-line of the form described in (3.4).

The divergence of the vector field $\vec{F}(F_1, F_2)$ becomes zero in the first quadrant of the phase plane $(O, w(t), u(t))$ along the half-line (3.4). Suppose that this half-line also includes the point $M(w_4, u_4)$, where $w(t) > 0$.

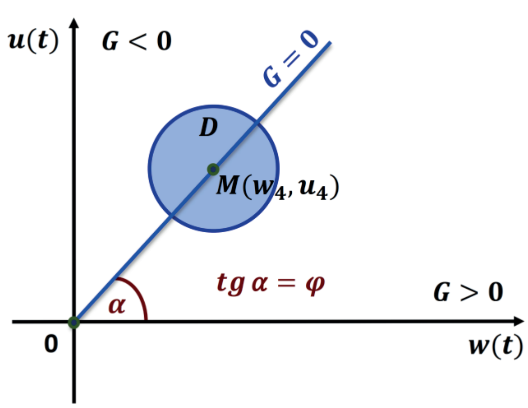

Fig. 3.1

It is evident that the divergence of the vector field $\vec{F}(F_1, F_2)$, denoted as $G(w(t), u(t))$ in (3.2), changes sign in some simply connected domain $D \subset (O, w(t), u(t))$ that contains the point $M(w_4, u_4)$ (see Fig. 3.1).

According to the Poincaré-Bendixson theorem [15 - 18], for the dynamic system defined by equations (2.13), (2.14), (2.3), (2.5), (2.10), and (3.1), there exists a closed integral curve that is entirely contained within this domain.

Thus, **Theorem 3.1 is proved.**

Second Case. Suppose the following condition is satisfied:

$$\begin{cases} (\alpha_{10}-\delta_6)+(\alpha_{11}-\delta_7)=0 \\ \gamma_{17}-4\gamma_{20}>0 \\ \frac{\gamma_{18}}{\gamma_{15}}=\frac{3\gamma_{19}}{p(\gamma_{17}-4\gamma_{20})} \end{cases}, \tag{3.5}$$

which does not contradict the previously established conditions (2.3), (2.5), and (2.10). Then, taking into account (2.15), from (2.16) we obtain:

$$G(w,u)=\gamma_{15}w(t)-\gamma_{18}u(t)+3q\gamma_{19}\frac{w^2(t)}{u(t)}+(4\gamma_{20}-\gamma_{17})pq\frac{w^3(t)}{u^2(t)}=$$

$$=\gamma_{15}w(t)-\gamma_{18}u(t)+3q\gamma_{19}\frac{w^2(t)}{u(t)}-3q\frac{\gamma_{15}\gamma_{19}}{\gamma_{18}}\frac{w^3(t)}{u^2(t)}=$$

$$=\gamma_{15}\left[w(t)-\frac{\gamma_{18}}{\gamma_{15}}u(t)\right]-3q\frac{\gamma_{15}\gamma_{19}}{\gamma_{18}}\frac{w^2(t)}{u^2(t)}\left[w(t)-\frac{\gamma_{18}}{\gamma_{15}}u(t)\right]=$$

$$=\gamma_{15}\left[w(t)-\frac{\gamma_{18}}{\gamma_{15}}u(t)\right]\left[1-3q\frac{\gamma_{19}}{\gamma_{18}}\frac{w^2(t)}{u^2(t)}\right]=0. \tag{3.6}$$

For (3.6), we consider two subcases.

First Subcase. For equation (3.6), let us first consider the case where

$$w(t)=\frac{\gamma_{18}}{\gamma_{15}}u(t). \tag{3.7}$$

The following theorem holds.

Theorem 3.2.1. The problem (2.13), (2.14), (2.3), (2.5), (2.10), (3.5), (3.7) has a solution in the form of a closed integral trajectory within some simply connected domain $D\subset\big(O,w(t),u(t)\big)$ in the first quadrant of the phase plane of solutions $\big(O,w(t),u(t)\big)$ with physical significance, where the solution is entirely contained.

Proof. Consider a curve located on the phase plane $\big(O,w(t),u(t)\big)$, where the divergence of the vector field $\vec{F}(F_1,F_2)$ is equal to zero. Taking into account equations (2.15), (2.16), and (3.5), one of the solution curves of (3.6) will be a half-line of the form described in (3.7).

The divergence of the vector field $\vec{F}(F_1,F_2)$ becomes zero in the first quadrant of the phase plane $\big(O,w(t),u(t)\big)$ along the half-line (3.7). Suppose that this half-line also includes the point $M(w_4,u_4)$, where $w(t)>0$.

It is evident that the divergence of the vector field $\vec{F}(F_1,F_2)$, denoted as $G\big(w(t),u(t)\big)$ in (2.15), changes sign in some simply connected domain $D\subset\big(O,w(t),u(t)\big)$ that contains the point $M(w_4,u_4)$.

According to the Poincaré-Bendixson theorem [15 - 18], for the dynamic system defined by equations (2.13), (2.14), (2.3), (2.5), (2.10), (3.5), and (3.7), there exists a closed integral curve that is entirely contained within this domain.

Thus, **Theorem 3.2.1 is proved**.

Second Subcase. For equation (3.6), consider the second case where

$$w(t) = \sqrt{\frac{\gamma_{18}}{3q\gamma_{19}}}u(t), \tag{3.8}$$

then the following theorem is proved.

Theorem 3.2.2. The problem (2.13), (2.14), (2.3), (2.5), (2.10), (3.5), (3.8) has a solution in the form of a closed integral trajectory within some simply connected domain $D \subset \big(O, w(t), u(t)\big)$ in the first quadrant of the phase plane of solutions $\big(O, w(t), u(t)\big)$ with physical significance, where the solution is entirely contained.

Proof. Consider a curve located on the phase plane $\big(O, w(t), u(t)\big)$, where the divergence of the vector field $\vec{F}(F_1, F_2)$ is equal to zero. Taking into account equations (2.15), (2.16), and (3.5), the second solution curve of (3.6) will be a half-line of the form described in (3.8).

The divergence of the vector field $\vec{F}(F_1, F_2)$ becomes zero in the first quadrant of the phase plane $\big(O, w(t), u(t)\big)$ along the half-line (3.8). Suppose that this half-line also includes the point $M(w_4, u_4)$, where $w(t) > 0$.

It is evident that the divergence of the vector field $\vec{F}(F_1, F_2)$, denoted as $G\big(w(t), u(t)\big)$ in (2.15), changes sign in some simply connected domain $D \subset \big(O, w(t), u(t)\big)$ that contains the point $M(w_4, u_4)$.

According to the Poincaré-Bendixson theorem [15 - 18], for the dynamic system defined by equations (2.13), (2.14), (2.3), (2.5), (2.10), (3.5), and (3.8), there exists a closed integral curve that is entirely contained within this domain.

Thus, **Theorem 3.2.2 is proved**.

Third Case. Suppose the following condition is satisfied:

$$\begin{cases} (\alpha_{10} - \delta_6) + (\alpha_{11} - \delta_7) = 0 \\ 4\gamma_{20} - \gamma_{17} \neq 0 \end{cases}, \tag{3.9}$$

which does not contradict the previously established conditions (2.3), (2.5), and (2.10). Then, taking into account (2.15), from (2.16) we obtain:

$$G(w, u) = \gamma_{15}w(t) - \gamma_{18}u(t) + 3q\gamma_{19}\frac{w^2(t)}{u(t)} + (4\gamma_{20} - \gamma_{17})pq\frac{w^3(t)}{u^2(t)} = 0. \tag{3.10}$$

In the first quadrant of the phase plane $\big(O, w(t), u(t)\big)$, which has physical significance, we seek a solution to (3.10) in the following form:

$$w(t) = xu(t), \tag{3.11}$$

where $u > 0, w > 0, x > 0$, then, from (3.10), we obtain:

$$G(xu, u) = \gamma_{15}xu(t) - \gamma_{18}u(t) + 3q\gamma_{19}x^2u(t) + (4\gamma_{20} - \gamma_{17})pqx^3u(t) =$$
$$= u(t)[(4\gamma_{20} - \gamma_{17})pqx^3 + 3q\gamma_{19}x^2 + \gamma_{15}x - \gamma_{18}] = u(t)f(x),$$

where

$$f(x) \equiv (4\gamma_{20}-\gamma_{17})pqx^3+3q\gamma_{19}x^2+\gamma_{15}x-\gamma_{18}. \tag{3.12}$$

We now consider the cubic function (3.12).

First Subcase. Suppose that

$$4\gamma_{20}-\gamma_{17}>0, \tag{3.13}$$

then, for the cubic function $f(x)$, the following holds:

$$f(0)=-\gamma_{18}<0, f(-\infty)=-\infty, f(+\infty)=+\infty. \tag{3.14}$$

According to the Bolzano-Cauchy theorem, since the continuous function $f(x)$ takes values of opposite signs on the interval $(-\infty,+\infty)$, it must have at least one zero in this interval. Furthermore,

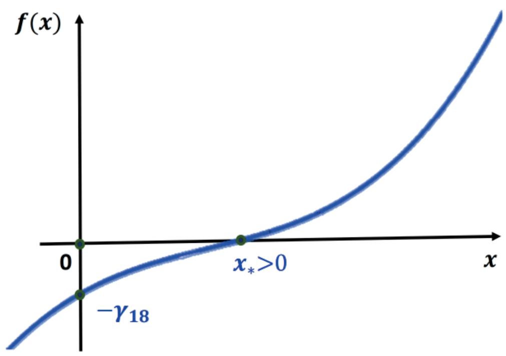

Fig. 3.3.1

$$f'(x)=3(4\gamma_{20}-\gamma_{17})pqx^2+6q\gamma_{19}x+\gamma_{15} > 0,$$

$$f''(x)=6(4\gamma_{20}-\gamma_{17})pqx+6q\gamma_{19}>0,$$

implying that the function $f(x)$ is increasing for positive x, and thus, it has no inflection points.

Consequently, the equation $f(x)=0$ has exactly one positive root x_* (see Fig. 3.3.1).

Consider the cubic equation

$$(4\gamma_{20}-\gamma_{17})pqx^3+3q\gamma_{19}x^2+\gamma_{15}x-\gamma_{18}=0 \Leftrightarrow$$

$$x^3+\frac{3\gamma_{19}}{(4\gamma_{20}-\gamma_{17})p}x^2+\frac{\gamma_{15}}{(4\gamma_{20}-\gamma_{17})pq}x-\frac{\gamma_{18}}{(4\gamma_{20}-\gamma_{17})pq}=0. \tag{3.15}$$

Using the Tartaglia transformation

$$x=y-\frac{\gamma_{19}}{(4\gamma_{20}-\gamma_{17})p} \tag{3.16}$$

equation (3.15) is reduced to the following depressed cubic equation:

$$y^3+p_1y+q_1=0, \tag{3.17}$$

where

$$p_1=\frac{\gamma_{15}}{(4\gamma_{20}-\gamma_{17})pq}-\frac{3\gamma_{19}^2}{(4\gamma_{20}-\gamma_{17})^2p^2}, \tag{3.18}$$

$$q_1=-\frac{\gamma_{18}}{(4\gamma_{20}-\gamma_{17})pq}+\frac{2\gamma_{19}^3}{(4\gamma_{20}-\gamma_{17})^3p^3}-\frac{\gamma_{15}\gamma_{19}}{(4\gamma_{20}-\gamma_{17})^2p^2q}.$$

Equation (3.17) has exactly one positive root, which is computed using the Ferro-Tartaglia-Cardano formula:

$$y=\sqrt[3]{-\frac{q_1}{2}+\sqrt{\frac{q_1^2}{4}+\frac{p_1^3}{27}}}+\sqrt[3]{-\frac{q_1}{2}-\sqrt{\frac{q_1^2}{4}+\frac{p_1^3}{27}}}, \tag{3.19}$$

Taking (3.19) into account, from (3.16), the value of x is obtained as:

$$x = \sqrt[3]{-\frac{q_1}{2} + \sqrt{\frac{q_1^2}{4} + \frac{p_1^3}{27}}} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{\frac{q_1^2}{4} + \frac{p_1^3}{27}}} - \frac{\gamma_{19}}{(4\gamma_{20}-\gamma_{17})p} > 0, \tag{3.20}$$

The following theorem holds.

Theorem 3.3.1. The problem (2.13), (2.14), (2.3), (2.5), (2.10), (3.9), (3.13) has a solution in the form of a closed integral trajectory within some simply connected domain $D \subset (O, w(t), u(t))$ in the first quadrant of the phase plane of solutions $(O, w(t), u(t))$ with physical significance, where the solution is entirely contained.

Proof. Consider a curve located on the phase plane $(O, w(t), u(t))$, where the divergence of the vector field $\vec{F}(F_1, F_2)$ is equal to zero. Taking into account equations (3.9) and (3.13), the curve (2.15) will be a half-line of the form described in (3.11).

The divergence of the vector field $\vec{F}(F_1, F_2)$ becomes zero in the first quadrant of the phase plane $(O, w(t), u(t))$ along the half-line (3.11). Suppose that this half-line also includes the point $M(w_4, u_4)$, where $w(t) > 0$.

It is evident that the divergence of the vector field $\vec{F}(F_1, F_2)$, denoted as $G(w(t), u(t))$ in (3.10), changes sign in some simply connected domain $D \subset (O, w(t), u(t))$ that contains the point $M(w_4, u_4)$.

According to the Poincaré-Bendixson theorem [15 - 18], for the dynamic system defined by equations (2.13), (2.14), (2.3), (2.5), (2.10), (3.9), and (3.13), there exists a closed integral curve that is entirely contained within this domain.

Thus, **Theorem 3.3.1 is proved.**

Second Subcase. When the condition

$$4\gamma_{20} - \gamma_{17} < 0, \tag{3.21}$$

is satisfied, then for the function $f(x)$ in (3.12), the following holds:

$$f(-\infty) = +\infty, f(+\infty) = -\infty. \tag{3.22}$$

In this case, the root x_* takes a negative value (see Fig. 3.3.2), which contradicts the condition defined by (3.11), according to which the point x_* must be positive.

Thus, during the fourth period, influenced by state governance and cultural factors, the number of Georgian-speaking populations significantly increases. The Georgian language emerges as one of the leading languages in the Caucasus, resulting in its complete dominance as both a state and religious language. Other Kartvelian languages − Laz, Mingrelian, and Svan − remain relatively balanced, as they retain their linguistic distinctiveness, which, despite assimilation processes, ensures their cultural space. The Georgian language exerts a powerful influence on these languages, further intensifying assimilation processes. Nevertheless, populations speaking all four languages ($u(t) > 0$, $w(t) > 0$, $z(t) > 0$, $v(t) > 0$) continue to coexist, despite the Georgian language's full dominance in the spheres of state, religious, and cultural interactions.

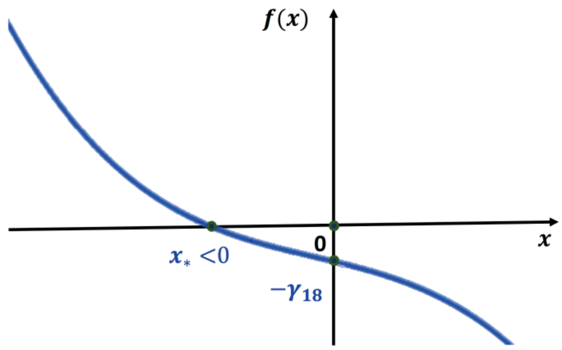

Fig. 3.3.2

Conclusion

In the fourth period (from the 1st century BC to the present), mathematical models describing the dynamics of populations speaking Georgian, Laz, Mingrelian, and Svan languages are formulated as a Cauchy problem for a four-dimensional nonlinear dynamic system with variable coefficients (a system of four equations with four unknowns). By employing first integrals, this constant-coefficient dynamic system was reduced to a two-dimensional system, focusing on the dynamics of the Georgian and Laz populations, while the dynamics of the Mingrelian and Svan populations are expressed through the demographic data of the Georgian and Laz populations. Using the Poincaré-Bendixson criterion, the existence of bounded integral trajectories in the physically meaningful first quadrant of the phase plane of the dynamic system's solutions is proven in four theorems for various cases, within certain simply connected regions. This demonstrates that the growth of the Georgian population leads to a decline in the Laz, Mingrelian, and Svan populations. However, the stability of the collective ethnolinguistic diversity, cultural identity, and the strengthening and dominance of the Georgian language are also established. These populations stably coexist within the same region.

Thus, mathematical models and corresponding computer simulations serve as powerful and unique tools that effectively describe the demographic changes of the proto-Kartvelian populations and the evolution of Kartvelian languages – Georgian, Laz, Mingrelian, and Svan – as well as the preservation of their cultural identity and stable coexistence.

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წარმოადგინა ცხუმ-აფხაზეთის მეცნიერებათა აკადემიის აკადემიკოს ილია ვეკუას სახელობის მათემატიკის ინსტიტუტმა

აბსტრაქტი. ნაშრომში წარმოდგენილია ქართველური მოსახლეობის ტრანსფორმაციის მეოთხე პერიოდი, რომლის პირველი მეოთხედის შემდეგ კოლხურ (ზანურ) ენიდან წარმოიშვა ლაზური და მეგრული. საბოლოოდ აღნიშნულ ეტაპზე ჩამოყალიბდა და განვითარდა ლინგვისტურად დადასტურებული ოთხი ქართველური ენები: ქართული, ლაზური, მეგრული და სვანური. ეს პროცესი დაკავშირებულია მათ შორის მიმდინარე კულტურულ-ენობრივ ურთიერთქმედებებთან, ისე გარე ფაქტორების ზეგავლენებთან, მათ შორის ინდოევროპული და ნაწილობრივ სემიტური ეთნიკური ჯგუფების ზემოქმედებებთან, როგორიცაა ასიმილაცია თუ დისიმილაცია.

ქართველურ ენებზე მეტყველ მოსახლეობათა შორის ურთიერთქმედებისა და მათი დემოგრაფიული ტრანსფორმაციის პროცესები და რაოდენობრივი დინამიკა აღიწერება მათემატიკური მოდელით, რომელიც ჩაწერილია ოთხგანზომილებიანი (ოთხუცნობიანი) ცვლადკოეფიციენტებიანი არაწრფივი დინამიური სისტემის სახით, რომლისთვის კოშის ამოცანაა დასმული. სისტემის თითოეული განტოლება ასახავს კონკრეტული ქართველური ენობრივი ჯგუფის რაოდენობრივი ცვლილების პროცესის დროის ფუნქციას. მოდელში გათვალისწინებულია ამ მოსახლეობების შემდეგი ფაქტორები: დემოგრაფიული (ბუნებრივი შობადობა-სიკვდილიანობის) მაჩვენებლები, ენობრივ-კულტურული ურთიერთქმედებები (ასიმილაცია), ასევე გარე ფაქტორები (ომები, ეკოლოგიური კრიზისები, მიგრაციები, ეპიდემიები) და სხვა.

აღნიშნული მოდელი იძლევა ქართველურ ენებზე მოსაუბრე მოსახლეობათა ისტორიული განვითარების, ინტეგრაციისა და დიფერენციაციის ტენდენციების რაოდენობრივი ანალიზის შესაძლებლობას, რაც ქმნის საფუძველს მათი ევოლუციური პროცესების კომპლექსური შეფასებისთვის.

განხილულია აღნიშნული სისტემების პირველი ინტეგრალების პოვნის მეთოდები. კვლევა ეფუძნება ოთხგანზომილებიანი დინამიური სისტემის ანალიზს, რომელიც აღწერს ქართველური – ქართული, ლაზური, მეგრული და სვანური მოსახლეობების ენობრივ ტრანსფორმაციებსა და მათ შორის მიმდინარე ურთიერთქმედებებს დროში. ძირითადად ყურადღება გამახვილებულია დინამიური სისტემის შესწავლაზე მუდმივი კოეფიციენტების შემთხვევაში, რაც ადეკვატურია ქართველური მოსახლეობის ტრანსფორმაციის მეოთხე პერიოდის ანალიზისთვის.

აღნიშნული დინამიური სისტემა გარდაიქმნება ლოგარითმული სახით, რაც მის ანალიზს ეფექტურს ხდის. შესაბამისი კოეფიციენტებისთვის გარკვეული პირობების დაკმაყოფილების შემთხვევაში, შესაძლებელია ოთხ- და შემდეგ სამგანზომილებიანი დინამიური სისტემების პირველი ინტეგრალების პოვნა, რომლებიც აღწერს უცნობი ფუნქციების ურთიერთდამოკიდებულებას ოთხგანზომილებიან ჰიპერსივრცეში. ორი პირველი ინტეგრალის გამოყენებით ოთხგანზომილებიანი არაწრფივი სისტემა დაიყვანება ორგანზომილებიან

სისტემაზე, რომელიც მხოლოდ ქართული და ლაზური ჯგუფის დინამიკას აღწერს. მეგრულენოვნისა და სვანურენოვანის რაოდენობები დაკავშირებულია ქართულენოვნისა და ლაზურენოვანის რაოდენობებთან პირველი ინტეგრალების ამსახველი ალგებრული თანაფარდობებით. ვექტორული ველის დივერგენციის ნულის ტოლობის პირობის დროს სხვადასხვა შემთხვევაში ორგანზომილებიანი დინამიური სისტემა გამოკვლეულია.

დამტკიცებულია თეორემები ქართული, ლაზური, მეგრული და სვანური მოსახლეობების თანაცხოვრობის შესახებ. მასში შესწავლილია მათემატიკური მოდელი, რომელიც, ორგანზომილებიანი დინამიური სისტემის სახით, აღწერს ქართველური – ქართული და ლაზური მოსახლეობების ენობრივ ტრანსფორმაციებსა და დემოგრაფიულ ცვლილებებს. განსაზღვრულია ქართული, ლაზური, მეგრული და სვანური მოსახლეობების თანაცხოვრების დინამიკის პირობები სხვადასხვა სცენარისთვის, რისთვისაც მათემატიკურად ფორმულირებული და დამტკიცებულია ოთხი თეორემა. აღნიშნული თეორემების დასამტკიცებლად გამოყენებულია პუანკარე-ბენდიქსონის პრინციპი, რომელიც ადასტურებს შეკრული ინტეგრალური ტრაექტორიების არსებობას ორგანზომილებიანი დინამიურ სისტემისთვის. ამით ნაჩვენებია, რომ არცერთი ქართველური ენობრივი ჯგუფი არ გაქრება. ოთხივე ინარჩუნებს თავიანთ არსებობას განსაზღვრული პირობების ფარგლებში.

საკვანძო სიტყვები: *მათემატიკური მოდელირება, პროტო-ქართველური მოსახლეობის ტრანსფორმაცია, ქართული, ლაზური, მეგრული და სვანური მოსახლეობები და ენები, მათემატიკური მოდელები, ლოტკა-ვოლტერას სისტემა, ანალიტიკური ამონახსნი, პირველი ინტეგრალები, ოთხგანზომილებიანი ჰიპერსივრცე, პუანკარე-ბენდიქსონის თეორემა.*