ON HYPERDETERMINANTS AND EQUATIONS OVER NONCOMMUTATIVE RINGS

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Abstract. The main goal of the paper is to investigate some features of polynomials and hyperdeterminants over noncommutative rings, namely over quaternion skew-field and division rings with involution; these results generalized well-known results of A. Cayley, I. Gelfand, M. Kapranov, A. Zelevinsky, X. Zhao, Y. Zhang and others. Main results are: the estimation of number of roots of canonical polynomials over quaternions – they are infinite unlike number of roots of polynomials over real field and generalization of results of X. Zhao, Y. Zhang on resultants and its features of polynomials over quaternions to polynomials over division rings with involution (Theorems 2-5). Also, in last paragraph of the paper is hypothesized what form should it be the cubical hyperdeterminant of order three over division ring with involution.

Keywords: *Quaternions, resultant, noncommutative hyperdeterminants, division rings with involution.*

Introduction. The estimating the number of roots of polynomials, the notions of resultant and determinant are closely related to each other. This connection is also preserved in the generalizations of these studies and concepts, which is clearly seen in this work, and therefore these tasks are brought together here.

The work consists of three paragraphs. In the first it is present the theorem of N. Topuridze about the structure of roots of canonical polynomials over quaternions (Theorem 1), namely it is shown, that the zero-set of a canonical quaternion polynomial consists of t isolated points and s two-dimensional spheres, where $t + 2s$ does not exceed the algebraic degree of a given polynomial. X. Zhao, Y. Zhang [1] generalized resultant and Cramer law for quaternions;

in the second paragraph we generalize these results then the ground ring is a ring with division and involution. In the third paragraph based on results of A. Cayley [2], I. Gelfand, M. Kapranov, A. Zelevinsky [3] and others we hypothesize what form should it be the cubical hyperdeterminant of order three over division rings with involution.

1. Quaternion polynomials. Let H be the quaternion skew-field, i.e. a four-dimensional vector space R^4 over the field of real numbers R:

$$H = \{(a, b, c, d) |\ a, b, c, d \in R\}\,.$$

We denote the generators (unit vectors) by

$$1 = (1,0,0,0), \qquad i = (0,1,0,0), \qquad j = (0,0,1,0), \qquad k = (0,0,0,1).$$

Then any quaternion can be written in the form $a + bi + cj + dk$, where $a, b, c, d \in R$. The unit vectors i, j, and k are sometimes called imaginary units. The multiplication in H is defined by the famous rules found by R. Hamilton:

$$ij = -ji = k, jk = -kj = i, ki = -ik = j. \tag{1}$$

Quaternions form a system with division, i.e., the equations

$$\alpha q = \beta, \qquad q\alpha = \beta,$$

where q is an unknown quaternion, possess the solutions $q_l = \alpha^{-1}\beta$ and $q_r = \beta\alpha^{-1}$, respectively. Moreover, the quaternion norm is multiplicative, i.e., $Nr(\alpha\beta) = Nr(\alpha)Nr(\beta)$, where

$$Nr(\alpha) = a^2 + b^2 + c^2 + d^2.$$

Every quaternion satisfies a polynomial equation with real coefficients. More precisely, it can be verified directly that the quaternion $\alpha = a + bi + cj + dk$ satisfies the quadratic equation with real coefficients:

$$q^2 - 2aq + a^2 + b^2 + c^2 + d^2 = q^2 - 2Re(\alpha)q + Nr(\alpha) = 0.$$

The polynomial

$$f_\alpha(q) = q^2 - 2Re(\alpha)q + Nr(\alpha) \tag{2}$$

is called the characteristic polynomial of the quaternion α and is an irreducible quadratic trinomial from the ring of polynomials $R[q]$. The converse statement is also valid: if $g(q) = q^2 + 2tq + s$ is a quadratic trinomial with a negative discriminant, then any quaternion $\beta = a' + b'i + c'j + d'k$, for which $a' = Re(\beta) = -t,$ and $Nr(\beta) = s,$ is a root of the polynomial $g(q)$.

Thus there are infinitely many quaternions that are roots of such a quadratic trinomial, and it will be shown below that the roots of polynomial (2) form a two-dimensional sphere S^2. Thus, unlike the well-known situation in a field, where an nth degree polynomial may have not more than n roots in virtue of the Bezout theorem, a polynomial over H may have infinitely many roots.

Let us consider the polynomial of one variable with coefficients from H

$$P(q) = \sum_{m=0}^{n} \xi_m q^m, \text{ where } \xi_i \epsilon H \text{ and } \xi_n = 1.$$

Theorem 1. *The zero-set of a canonical quaternion polynomial*

$$P(q) = \sum_{m=0}^{n} \xi_m q^m,\ \alpha_i \in H, i = \overline{0,n},\ a_n = 1$$

consists of t *isolated points and* $s \le \frac{n-t}{2}$ *two-dimensional spheres, i.e., the inequality* $t + 2s \le n$ *is valid.*

As is shown, the set of roots of a canonical polynomial always consists of a finite number of points and two-dimensional spheres.

2. Resultants over noncommutative rings. Let K be an associative division ring with involution $\alpha \to \bar{\alpha}, \bar{\bar{\alpha}} = \alpha, \alpha \in K$. Quaternions are the example of such rings.

Let $K[x]$ be canonical polynomials (i.e. the coefficients and variables are separated and the coefficients are at the beginning) over K:

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n.$$

Let $gcrd(f,g)$ denotes the greatest right common divisor of polynomials $f, g \in K[x]$.

Sylvester matrix $Syl(f,g)$ [1] of two canonical polynomials

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m, \qquad g(x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n$$

is the matrix

$$\begin{pmatrix}
a_m & & & & b_n & & & \\
a_{m-1} & a_m & & & b_{n-1} & b_n & & \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\
\vdots & \vdots & & a_m & \vdots & \vdots & & \ddots \\
\vdots & \vdots & & a_{m-1} & b_0 & \vdots & & b_n \\
a_0 & \vdots & & \vdots & & b_0 & & \vdots \\
 & a_0 & & \vdots & & & \ddots & \vdots \\
 & & \ddots & \vdots & & & \ddots & \vdots \\
 & & & a_0 & & & & b_0
\end{pmatrix}$$

with n columns of a_i-s and m columns of b_j-s, and all entries outside the two "parallelograms" are zero. The double determinant [4] of the transpose of Sylvester matrix is called [1] the resultant of f and g, denoted by $res(f,g) = ddet(Syl(f,g)^T)$, where $Syl(f,g)^T$ is the transpose of the Sylvester matrix $Syl(f,g)$. Recall that the double determinant of a matrix A over ring K is the raw determinant [4] $rdet_i(A^*A)$ (here A^* denotes the involutive and transpose matrix of A) which does not depends from the choose of the raw (column) i because A^*A is Hermitian.

Similarly as in [1] we prove theorems 1- 4:

Theorem 1. *(Cramer's Rule). Let* $xA = y$ *be a left system of linear equation over* K *with coefficient matrix* A*, constant row* $y = (y_1, y_2, \dots, y_n)$ *of elements from over* K*, and unknowns* $x = (x_1, x_2, \dots, x_n)$*. If double determinant* $ddet(A) \neq 0$*, then the system has a unique solution in* K *given by*

$$x_i = \frac{rdet_i((AA^*)_{i.}(yA^*))}{ddet(A)}, \quad 1 \leq i \leq n,$$

where $(AA^*)_{i.}(yA^*)$ *is the matrix obtained from* AA^* *by replacing the i-th row by the row vector* yA^*.

Theorem 2. *Let* K *be an associative division ring with involution. Suppose* $f, g \in K[x]$ *are nonzero. Then* $gcrd(f, g) = 1$ *if and only if* $res(f, g) \neq 0$.

Theorem 3. *Let* K *be an associative division ring with involution. Suppose* $f, g \in K[x]$ *and* $deg\ f > 0$*,* $deg\ g > 0$*. Then there exist polynomials* $p, q \in K[x]$ *such that* $pf + qg = res(f, g)$*. Furthermore the coefficients of* p *and* q *are integer polynomials in the coefficients of* f *and* g.

Theorem 4. *Let* $0 \neq f \in K[x]$*. Then* f *has a repeated right root if and only if* $res(f, f') = 0$.

From these theorems follows

Theorem 5. *Let* K *be an associative division ring with involution. Suppose* $f, g \in K[x]$ *and* $deg\ f > 0$*,* $deg\ g > 0$*. Then* f, g *have common roots if and only if* $res(f, g) = 0$.

Indeed if $f, g \in K[x]$, $deg\, f > 0$, $deg\ g > 0$ and f, g have common root α then $res(f, g) = 0$. From Theorem 3 follows that

$$pf(\alpha) + qg(\alpha) = 0 = res(f, g).$$

Conversely if $f, g \in K[x]$, $deg\ f > 0$, $deg\ g > 0$ and $res(f, g) = 0$, then f, g have common root. Indeed since $res(f, g) = 0$, by Theorem 2 $gcrd(f, g) = d(x)$, $\deg d(x) > 1$ and

$$f(x) = f_1(x)d(x),\ g(x) = g_1(x)d(x).$$

If α is a root of $d(x)$, then it is clear that $f(\alpha) = g(\alpha) = 0$, i.e. $f(x)$ and $g(x)$ have common root.

3. Noncommutative hyperdeterminants. The determinant of a matrix one can extend by two ways to hyperdeterminants of higher order: a) by extending the usual expression of an $n \times n$ matrix determinant, which we will call the *combinatorial hyperdeterminant*, b) by using the characterization that a matrix has $\det A = 0$ if and only if $Ax = 0$ has nonzero solutions, which we will call the *geometric hyperdeterminant*. This approaches were proposed by Cayley [2], he also gave the explicit expression of a $2 \times 2 \times 2$ geometric hyperdeterminant.

The combinatorial hyperdeterminant of a cubical $A = (a_{i_1 i_2 \dots i_d}) \in F^{n\times \dots \times n}$ d-hypermatrix is

$$\det(A) = \frac{1}{n!}\sum_{\pi_1,\dots,\pi_d \in S_n} sgn\pi_1 \dots sgn\pi_d \prod_{i=1}^{n} a_{\pi_1(i)\dots\pi_d(i)}.$$

Particulary, for odd order d, the combinatorial hyperdeterminant of a cubical d-hypermatrix is identically zero and for even order d, the combinatorial hyperdeterminant of a cubical d-hypermatrix

$$\det(A) = \sum_{\pi_2,\dots,\pi_d \in S_n} sgn(\pi_2 \dots \pi_d) \prod_{i=1}^{n} a_{i\pi_2(i)\dots\pi_d(i)}.$$

Obtaining an explicit formula for the hyperdeterminant even for commutative rings is not an easy task. See, for example, [5] which shows that the 2×2×2×2 hyperdeterminant consists of 2,894,276 terms. We think that the new formula proposed in [6] (and a method to obtain) the 2×2×2 hyperdeterminant for commutative rings might be extendable to hyperdeterminants of noncommutative rings.

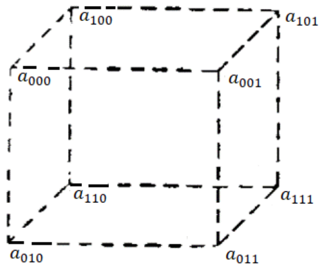
As is known [2] in the commutative case the discriminant of system of equations

$$\begin{gathered} a_{000}x_0y_0 + a_{010}x_0y_1 + a_{100}x_1y_0 + a_{110}x_1y_1 = 0 \\ a_{001}x_0y_0 + a_{001}x_0y_1 + a_{001}x_1y_0 + a_{001}x_1y_1 = 0 \\ a_{000}x_0y_0 + a_{001}x_0y_1 + a_{100}x_1y_0 + a_{101}x_1y_1 = 0 \\ a_{010}x_0y_0 + a_{011}x_0y_1 + a_{110}x_1y_0 + a_{111}x_1y_1 = 0 \\ a_{000}y_0z_0 + a_{001}y_0z_1 + a_{010}y_1z_0 + a_{011}y_1z_1 = 0 \\ a_{100}y_0z_0 + a_{100}y_0z_1 + a_{100}x_1y_0 + a_{111}x_1y_1 = 0 \end{gathered} \tag{3}$$

is a 2x2x2 geometric hyperdeterminant

$$\begin{gathered} -a_{000}^2a_{111}^2 - a_{100}^2a_{011}^2 - a_{010}^2a_{101}^2 - a_{001}^2a_{11}^2 \\ -4a_{000}a_{110}a_{101}a_{011} - 4a_{000}a_{110}a_{101}a_{011} \\ +2a_{000}a_{100}a_{011}a_{111} + 2a_{000}a_{010}a_{101}a_{111} \\ +2a_{000}a_{001}a_{110}a_{111} + 2a_{100}a_{010}a_{101}a_{011} \\ +2a_{100}a_{001}a_{110}a_{011} + 2a_{010}a_{001}a_{110}a_{101} \end{gathered} \tag{4}$$

of the cubical matrix



This means that (3) has nontrivial solutions if and only if (4) is zero. We can write the geometric hyperdeterminant of $A = (A_1|A_2) = (a_{ijk}) \in C^{2\times2\times2}$ (where A_1 and A_2 slices of A) also as [4]

$$Det_{2,2,2} = \frac{1}{4}\big(det(A_1 + A_2) - det(A_1 - A_2)\big)^2 - 4\det(A_1)\det(A_2)$$

$$= \frac{1}{4}\left(det\left(\begin{pmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{pmatrix} + \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix}\right) - det\left(\begin{pmatrix} a_{000} & a_{001} \\ a_{010} & a_{011} \end{pmatrix} - \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix}\right)\right)^2$$

$$-4det\begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} det\begin{pmatrix} a_{100} & a_{110} \\ a_{100} & a_{111} \end{pmatrix}.$$

The geometric hyperdeterminant of $A = (a_{ijk}) \in C^{2\times2\times3}$ is [4]

$$Det_{2,2,3} = det\begin{pmatrix} a_{000} & a_{001} & a_{002} \\ a_{100} & a_{101} & a_{102} \\ a_{010} & a_{011} & a_{012} \end{pmatrix} det\begin{pmatrix} a_{100} & a_{101} & a_{102} \\ a_{010} & a_{011} & a_{012} \\ a_{110} & a_{111} & a_{112} \end{pmatrix}$$

$$- det\begin{pmatrix} a_{000} & a_{001} & a_{002} \\ a_{100} & a_{101} & a_{102} \\ a_{110} & a_{111} & a_{112} \end{pmatrix} det\begin{pmatrix} a_{000} & a_{001} & a_{002} \\ a_{010} & a_{011} & a_{012} \\ a_{110} & a_{111} & a_{112} \end{pmatrix}.$$

The condition that $Det_{2,2,3} = 0$ is equivalent to statement, that the system

$$a_{000}x_0y_0 + a_{010}x_0y_1 + a_{100}x_1y_0 + a_{110}x_1y_1 = 0,$$
$$a_{001}x_0y_0 + a_{011}x_0y_1 + a_{101}x_1y_0 + a_{111}x_1y_1 = 0,$$
$$a_{002}x_0y_0 + a_{012}x_0y_1 + a_{102}x_1y_0 + a_{112}x_1y_1 = 0,$$
$$a_{000}x_0z_0 + a_{001}x_0z_1 + a_{002}x_0z_2 + a_{100}x_1z_0 + a_{101}x_1z_1 + a_{102}x_1z_2 = 0,$$
$$a_{010}x_0z_0 + a_{011}x_0z_1 + a_{012}x_0z_2 + a_{110}x_1z_0 + a_{111}x_1z_1 + a_{112}x_1z_2 = 0,$$
$$a_{000}y_0z_0 + a_{011}y_0z_1 + a_{012}y_0z_2 + a_{010}y_1z_0 + a_{011}y_1z_1 + a_{012}y_1z_2 = 0,$$
$$a_{100}y_0z_0 + a_{101}y_0z_1 + a_{102}y_0z_2 + a_{110}y_1z_0 + a_{111}y_1z_1 + a_{112}y_1z_2 = 0.$$

Our goal is to formulate the hypothesis, which generalizes the notion of *cubical geometric* 3-*hyperdeterminants* for associative division rings with involution.

Suppose (4) is the canonical equations over the associative division ring with involution K and

$$B_0 = \begin{pmatrix} a_{000} & a_{100} \\ a_{001} & a_{101} \end{pmatrix},\ B_1 = \begin{pmatrix} a_{010} & a_{110} \\ a_{011} & a_{111} \end{pmatrix},\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let A denotes the matrix [6]

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & {B_0}^T \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & {B_1}^T \\ B_0 & B_1 & 0 \end{pmatrix}$$

Now we can formulate our hypothesis:

Hypothesis. The discriminant of the system of equations (4) is the double determinant of the matrix A. More precisely (4) has nontrivial solution if and only if the double determinant of the matrix A is zero.

Let us remark that in suitable proposition from [6] is considered ordinary determinant instead of double determinant of noncommutative rings.

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არაკომუტაციურ რგოლებზე განმარტებული ჰიპერდეტერმინანტებისა და განტოლებების შესახებ

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წარმოადგინა ცხუმ-აფხაზეთის მეცნიერებათა აკადემიის ილია ვეკუას სახელობის მათემატიკის ინსტიტუტმა

აბსტრაქტი. მრავალწევრების ფესვების რაოდენობის შეფასება, რეზულტანტისა და დეტერმინანტის ცნებები მჭიდრო კავშირშია ერთმანეთთან. ეს კავშირი დაცულია ამ მათ განზოგადებებშიც, რაც ნათლად ჩანს ამ ნაშრომშიც და ამიტომაც არის ეს ამოცანები აქ ერთად წარმოდგენილი.

ნაშრომი შედგება სამი პარაგრაფისგან. პირველში მოყვანილია ნ. თოფურიძის თეორემა კვატერნიონებზე განსაზღვრული კანონიკური მრავალწევრების ფესვების აგებულების შესახებ (თეორემა 1), კერძოდ, ნაჩვენებია, რომ კანონიკური კვატერნიონული პოლინომის ფესვთა სიმრავლე შედგება t იზოლირებული წერტილისა და s ორგანზომილებიანი სფეროსგან, ამასთან $t+2s$ არ აღემატება მოცემული პოლინომის ხარისხს.

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