AN ANALYSIS OF THE HOPF BIFURCATION AND COMPUTER SIMULATION FOR ONE-DIMENSIONAL MAXWELL-TYPE NONLINEAR SYSTEM

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Abstract. One-dimensional nonlinear Maxwell-type system is considered. The initial-boundary value problem with mixed type boundary conditions is discussed. It is proved that in some cases of nonlinearity there exists critical value of the boundary data ψ_c, such that for a suffciently small positive values of ψ the steady state solution is linearly stable. But as ψ passes through a critical value ψ_c, the stability changes and a Hopf bifurcation may takes place. The finite difference scheme is constructed. Results of numerical experiments with graphical illustrations are given.

Keywords: *Maxwell-type one-dimensional nonlinear system, stationary solution, linear stability, Hopf-type bifurcation, finite difference scheme, computer simulation.*

Introduction

The present work deals with a nonlinear model which is obtained after adding two terms to the second equation of well-known Maxwell system in one-dimensional case (Landau, Lifschitz, 1957). This model is also some generalization of system describing many other processes (see, for example, (Dafermos, Hsiao, 1983, Jangveladze, 2019, Jangveladze, Kiguradze, Neta, 2016) and references therein).

The initial-boundary value problem with mixed type boundary conditions is considered for this model. It is proved that in some cases of nonlinearity there exists critical value ψ_c of the boundary data, such that for $0 < \psi < \psi_c$ the steady state solution of the studied problem is linearly stable, while for $\psi > \psi_c$ is unstable. It is shown that when ψ passes through ψ_c then the Hopf-type bifurcation may take place. The finite difference scheme is constructed. Results of numerical experiments are given.

Statement of the problem

In the cylinder $[0;1] \times [0;\infty)$ let's consider the following problem:

$$\begin{gathered}\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(V^\alpha \frac{\partial U}{\partial x}\right), \\ \frac{\partial V}{\partial t} = -aV^\beta + bV^\gamma \left(\frac{\partial U}{\partial x}\right)^2 + cV^\delta \frac{\partial U}{\partial x}, \\ U(0,t) = 0, V^\alpha \frac{\partial U}{\partial x}\bigg|_{x=1} = \psi, \\ U(x,0) = U_0(x), V(x,0) = V_0(x).\end{gathered} \tag{1}$$

Many works are dedicated to the investigation and numerical solution of (1) type models (see, for instance, (Dzhangveladze, 1987a, 1987b, 1989, Dzhangveladze, Lyubimov, Korshiya, 1986, Gagoshidze, Jangveladze, 2011, Jangveladze, 1999, 2014, 2019, Jangveladze, Gagoshidze, 2016, Jangveladze, Kratsashvili, 2018, Jangveladze, Kiguradze, Neta, 2016, Kiguradze, 2001). Here t and x are time and space variables respectively, $U = U(x;t), V = V(x;t)$ are unknown functions, U_0, V_0 are given functions, and $a.b.c.\alpha,\beta,\gamma,\delta,\psi$ are known positive parameters.

If $\delta = \gamma - \alpha$, it is easy to check that the unique stationary solution of problem (1) is

$$(U_s, V_s) = \left(\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{-\alpha}{2\alpha+\beta-\gamma}}\psi x,\ \left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{1}{2\alpha+\beta-\gamma}}\right). \tag{2}$$

Introducing a notation $W = V^\alpha \frac{\partial U}{\partial x}$, after simple transformations, we get

$$\begin{gathered}\frac{\partial W}{\partial t} = V^\alpha \frac{\partial^2 W}{\partial x^2} + \alpha\left(bV^{\gamma-2\alpha-1}W^2 + cV^{\gamma-2\alpha-1}W - aV^{\beta-1}\right)W, \\ \frac{\partial V}{\partial t} = -aV^\beta + bV^{\gamma-2\alpha}W^2 + cV^{\gamma-2\alpha}W,\end{gathered} \tag{3}$$

$$\left.\frac{\partial W}{\partial x}\right|_{x=0} = 0, W(1,t) = \psi,$$

$$W(x,0) = V_0^{\alpha}\frac{\partial U_0(x)}{\partial x}, V(x,0) = V_0(x).$$

If $2\alpha + \beta - \gamma \neq 0$ the stationary solution of problem (3) is

$$(W_s, V_s) = \left(\psi, \left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{1}{2\alpha+\beta-\gamma}}\right).$$

Linear stability of the stationary solution and Hopf bifurcation

We investigate the linear stability of problem (3) by linearizing near the stationary solution (Ws, Vs). Let

$$W(x,t) = W_s + W_1(x)e^{\lambda t} = \psi + W_1(x)e^{\lambda t},$$

$$V(x,t) = V_s + V_1(x)e^{\lambda t} = \left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{1}{2\alpha+\beta-\gamma}} + V_1(x)e^{\lambda t}.$$

After some transformations we have:

$$\frac{d^2W_1(x)}{dx^2} + \eta^2 W_1(x) = 0,$$
$$\left.\frac{dW_1(x)}{dx}\right|_{x=0} = W_1(1) = 0, \tag{4}$$

where

$$\eta^2 = \alpha a\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\beta-\alpha-1}{2\alpha+\beta-\gamma}} + b\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\gamma-3\alpha-1}{2\alpha+\beta-\gamma}}\psi^2$$
$$-\frac{a(2\alpha+\beta-\gamma)(2b\psi+c)\psi\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta+\gamma-3\alpha-2}{2\alpha+\beta-\gamma}}}{\lambda + a(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}} - \lambda\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{-\alpha}{2\alpha+\beta-\gamma}}.$$

It is not difficult to show that problem (4) has nontrivial solutions if and only if

$$\eta^2 = \eta_n^2 = \left(n + \frac{1}{2}\right)^2 \pi^2, n \in Z_0.$$

For corresponding $\lambda = \lambda_n$ we have

$$\lambda_n^2 - P_n(\psi, \alpha, \beta, \gamma, a, b, c)\lambda_n + L_n(\psi, \alpha, \beta, \gamma, a, b, c) = 0 \tag{5}$$

where:

$$P_n(\psi, \alpha, \beta, \gamma, a, b, c) = -\left(\eta^2\left(\frac{b}{a}\psi^2 + \frac{c}{a}\psi\right)^{\frac{\alpha}{2\alpha+\beta-\gamma}} + \right. \tag{6}$$

$$+A\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}-\alpha a\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-1}{2\alpha+\beta-\gamma}}-b\psi^2\Phi^{\frac{\gamma-2\alpha-1}{2\alpha+\beta-\gamma}}\Bigg),$$

$$L_n(\psi,\ \alpha,\beta,\gamma,a,b,c)=a(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\alpha+\beta-1}{2\alpha+\beta-\gamma}}-$$

$$-a(2\alpha+\beta-\gamma)(2b\psi+c)\psi\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta+\gamma-2\alpha-2}{2\alpha+\beta-\gamma}}-$$

$$-\alpha a^2(2\alpha+\beta-\gamma)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{2(\beta-1)}{2\alpha+\beta-\gamma}}-ba(2\alpha+\beta-\gamma)\psi^2\left(\frac{b}{a}\psi^2+\right.$$

$$\left.\frac{c}{a}\psi\right)^{\frac{\beta+\gamma-2\alpha-2}{2\alpha+\beta-\gamma}}.$$

The stationary solution $(Ws;\ Vs)$ of the problem (3) is linearly stabile only if $Re(\lambda_n)\ <\ 0$, for all n and unstable if there exists an integer m such that $Re(\lambda_m)\ >\ 0$.

From (5) and (6) it is obvious that if $2\alpha+\beta-\gamma>0$, stationary solution of problem (3) is linearly stable if and only if the inequalities hold:

$$P_n(\psi,\ \alpha,\beta,\gamma,a,b,c)<0, n\in Z_0.$$

From (5), (6) it can be deduced following statement.

Theorem. If $2\alpha+\beta-\gamma>0$, then stationary solution $(Ws;\ Vs)$ of problem (3) is linearly stable if and only if $P_n(\psi,\alpha,\beta,\gamma,a,b,c)<0$, for all n, i.e.,

$$a(\alpha-\beta+\gamma)\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\beta-\alpha-1}{2\alpha+\beta-\gamma}}+b\psi^2\left(\frac{b}{a}\psi^2+\frac{c}{a}\psi\right)^{\frac{\gamma-3\alpha-1}{2\alpha+\beta-\gamma}}<\frac{\pi^2}{4}.$$

We studied the stability of the steady state solution which depends on a boundary condition $\psi>\ 0$.

There exists critical value of the boundary data ψ_c, such that for $0<\ \psi<\ \psi_c$ the steady state solution of the studied problem is linearly stable, while for $\psi>\psi_c$ is unstable. So, as ψ passes through a critical value ψ_c, the stability changes and a Hopf bifurcation may takes place (Marsden, McCracken, 2012) .

The difference scheme

Let us enter grid $\overline{\omega}_{h\tau}=\overline{\omega}_h\times\omega_\tau$ on the area $\bar{Q}$=[0; 1] × [0; T], where T is the given positive constant, and:

$$\overline{\omega}_h=\{x_i=ih, i=0,\dots,\ N, hN=1\},$$

$$\omega_\tau=\{t_j=j\tau, j=0,\dots,M,\tau M=T\}.$$

Here h is a step in the x direction and τ is a time step in the interval [0, T].

Let us also use known notations:

$$y = y_i^j = y(x_i, t_j), \hat{y} = y_i^{j+1} = y(x_i, t_{j+1}),$$

$$y_t = \frac{\hat{y} - y}{\tau}, y_{\bar{x}} = \frac{y_i^j - y_{i-1}^j}{h}, y_x = \frac{y_{i+1}^j - y_i^j}{h},$$

$$y_{\bar{x}x} = \frac{y_{i+1}^j - 2y_i^j + y_{i-1}^j}{h^2},$$

and construct the following difference scheme for the problem (3):

$$w_t = \hat{v}^\alpha \hat{w}_{\bar{x}x} + \alpha\left(v^{\gamma-2\alpha-1} w^2 - v^{\beta-1}\right) w, \tag{7}$$

$$v_t = -av^\beta + bv^{\gamma-2\alpha} w^2 + cv^{\gamma-2\alpha} w, \tag{8}$$

$$w_{x,0} = 0, w_N = \psi, \tag{9}$$

$$w^0 = V_0^\alpha U_{0,x}, v^0 = V_0. \tag{10}$$

The difference scheme (7) - (10) is easily realizable, since to find an approximate solution of the layer $t = t_{j+1}$ it is necessary to solve in at first (8) for the function $\hat{v}$, and then the system of linear algebraic equations (7) for the function $\hat{w}$, using the known function $\hat{v}$. It should also be noted that the above algorithm does not require the use of an iterative process, since we find the function $\hat{v}$ explicitly from equation (8), and then the function $\hat{w}$ solving the system of linear algebraic equations with the threediagonal matrix.

Results of numerical experiments

Numerous numerical experiments have been performed using scheme (7) - (10) above. The results of the obtained experiments are consistent with theoretical studies. Below are some of them.

We use the test functions:

$$U(x, 0) = x^4\left((1 - x)^2 + \frac{\psi}{4}\right),$$

$$V(x, 0) = (sin\, \pi x)^4 + 2 - x,$$

$$W(x, 0) = V^\alpha(x, 0)\frac{\partial U(x, 0)}{\partial x},$$

and the following values of the parameters: $h = 0.01, \tau = 0.00004, N = 100, a = b = c = 1$.

Test 1. Case of convergence to the stationary solution: $\alpha = 1$, $\beta = 1$, $\gamma = 0$, $\psi = 1$, corresponding graphics are figures 1-8.

Test 2. Case of Hopf-type bifurcation: $\alpha = 1$, $\beta = 3$, $\gamma = 4$, $\psi = 1$, corresponding graphics are figures 9-13.

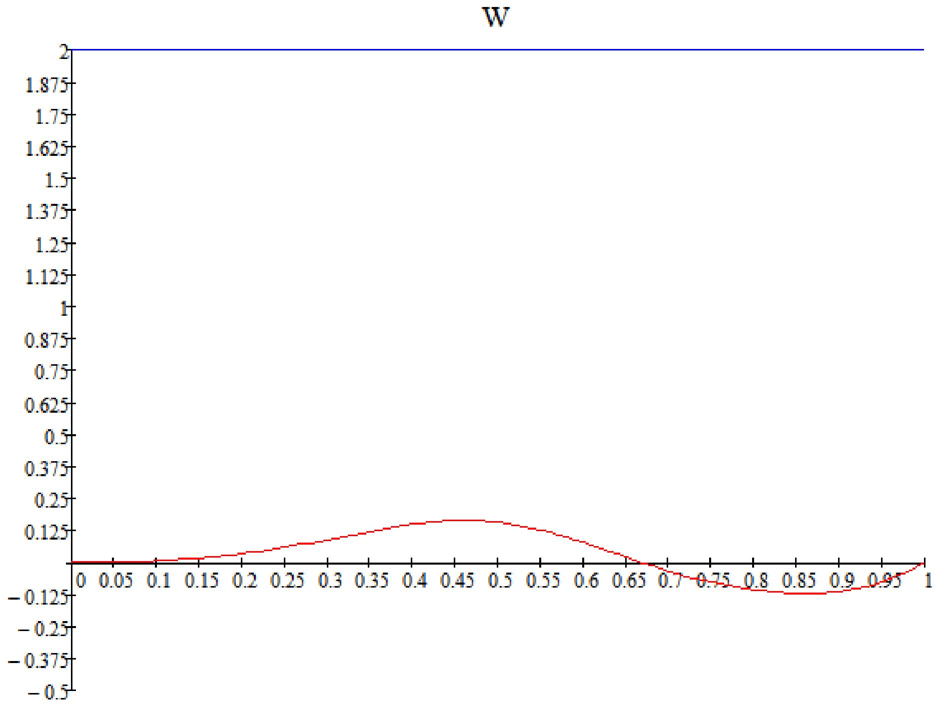

Fig. 1 ($t = 0$)

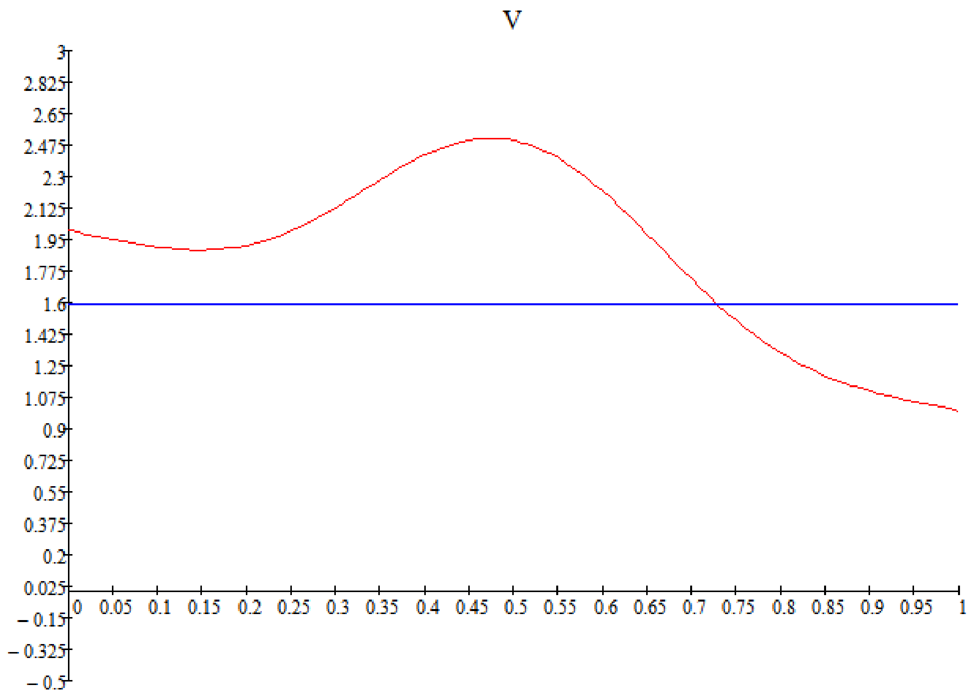

Fig. 2 ($t = 0$)

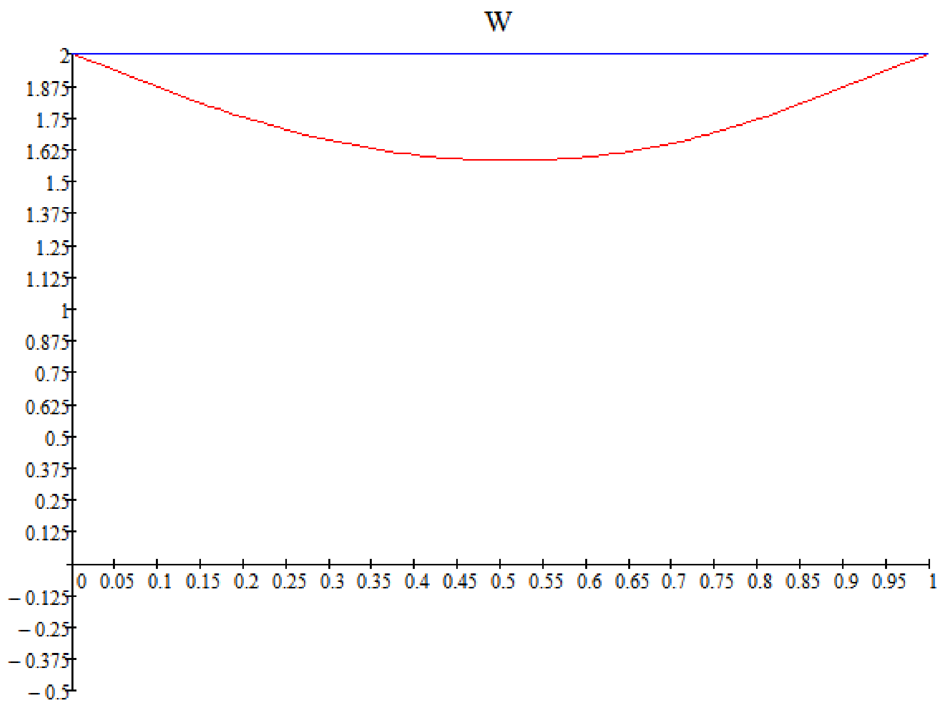

Fig. 3 ($t = 0.1$)

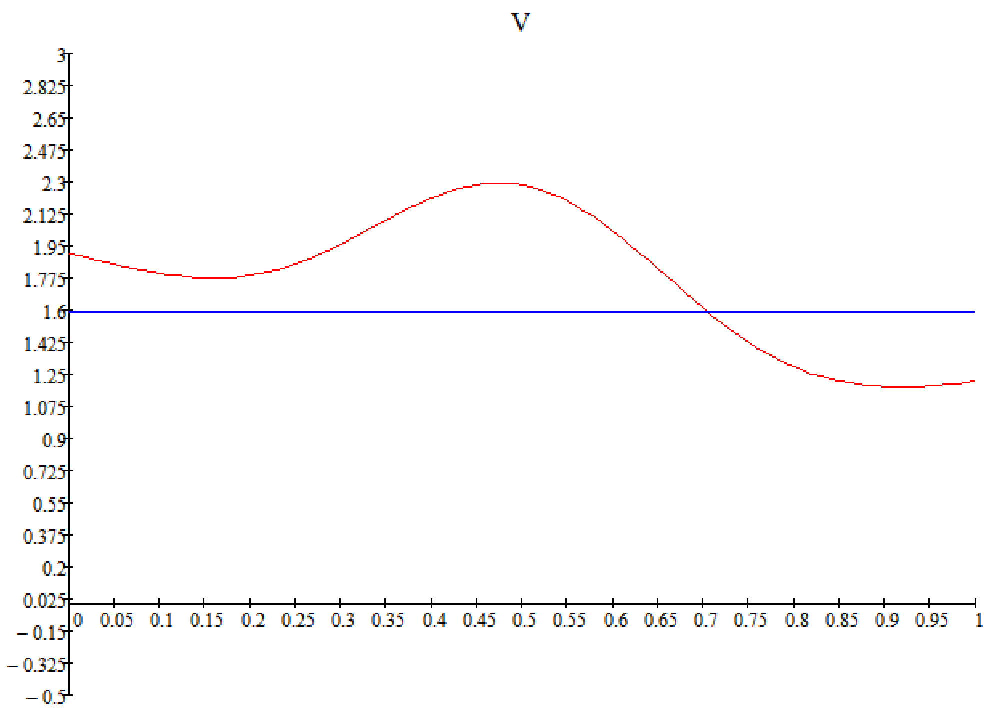

Fig. 4 ($t = 0.1$)

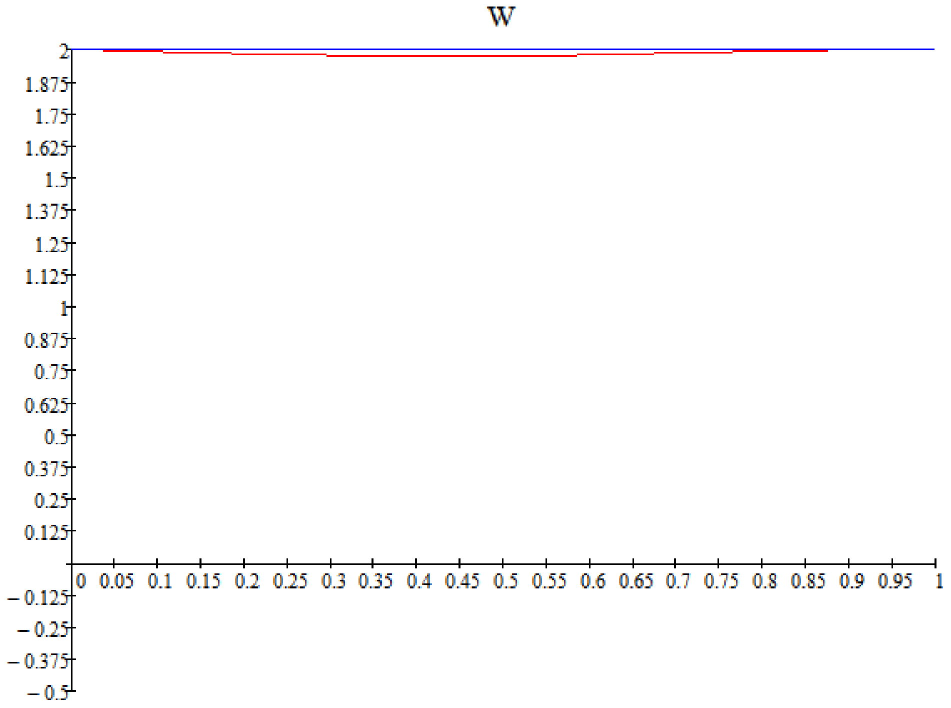

Fig. 5 ($t = 0.5$)

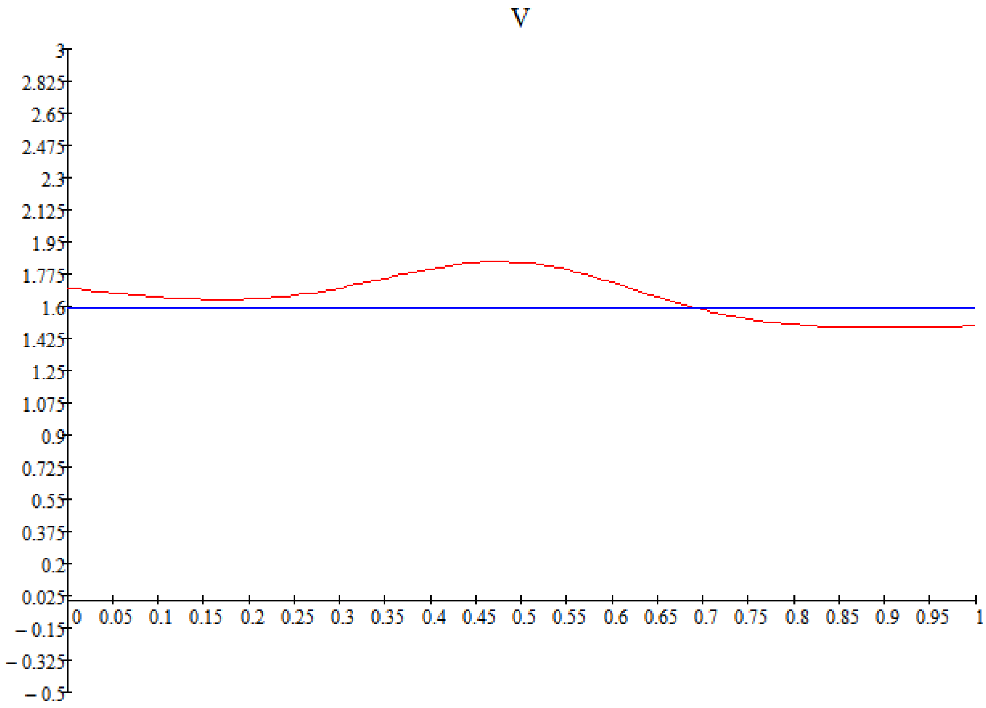

Fig. 6 ($t = 0.5$)

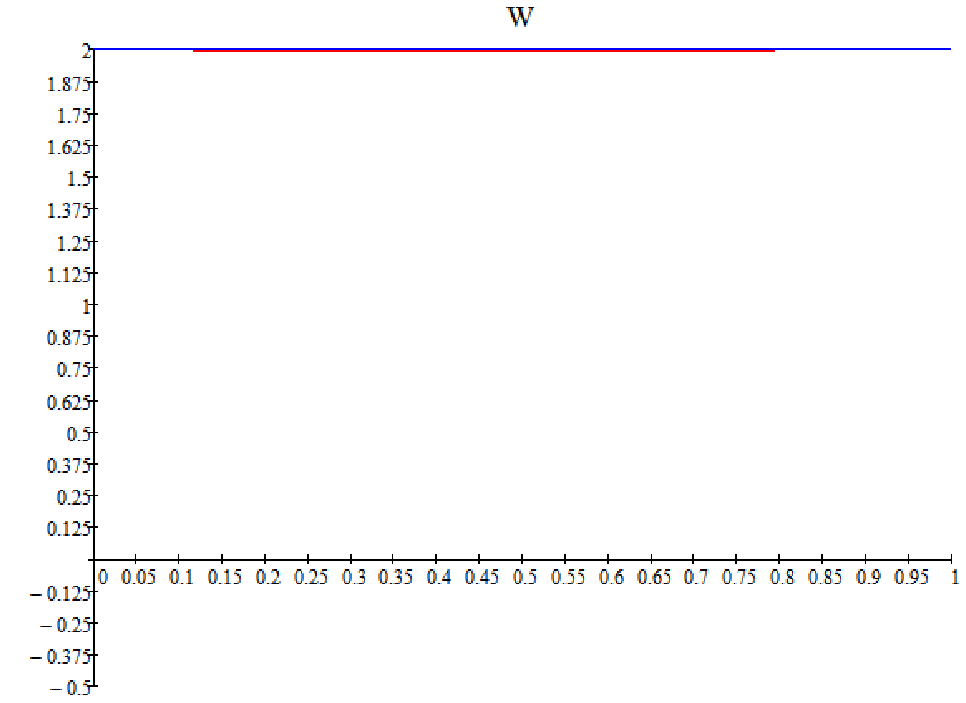

Fig. 7 ($t = 1$)

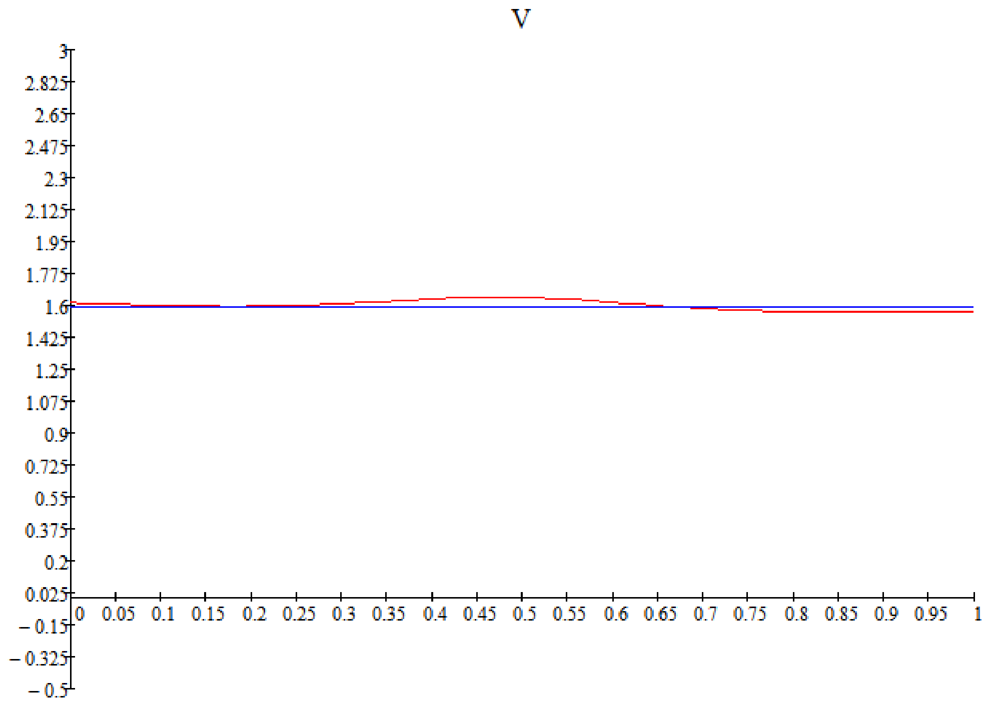

Fig. 8 ($t = 1$)

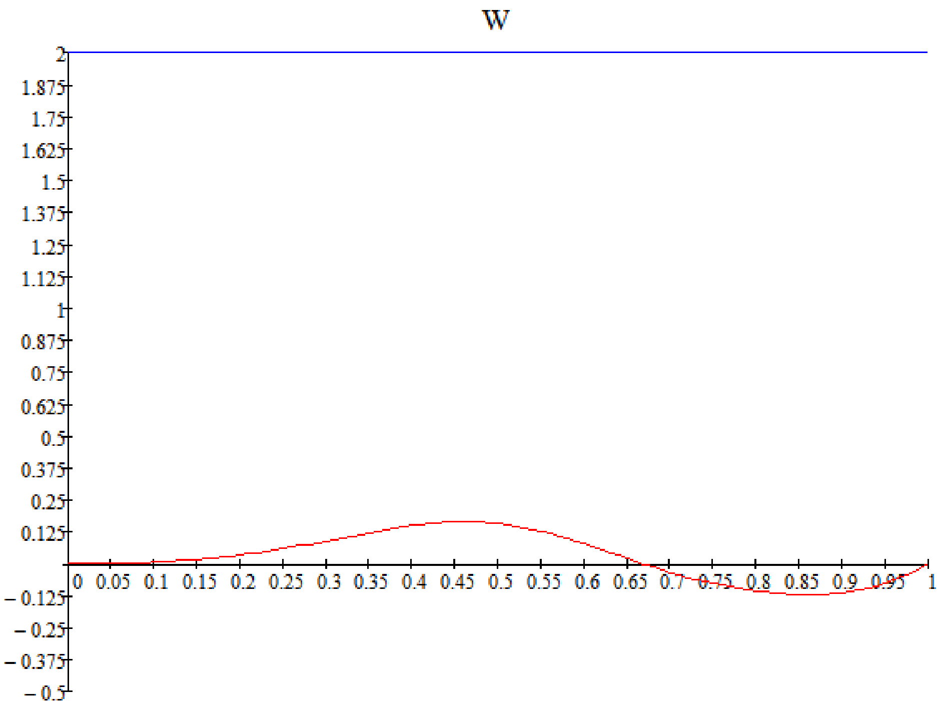

Fig. 9 ($t = 0$)

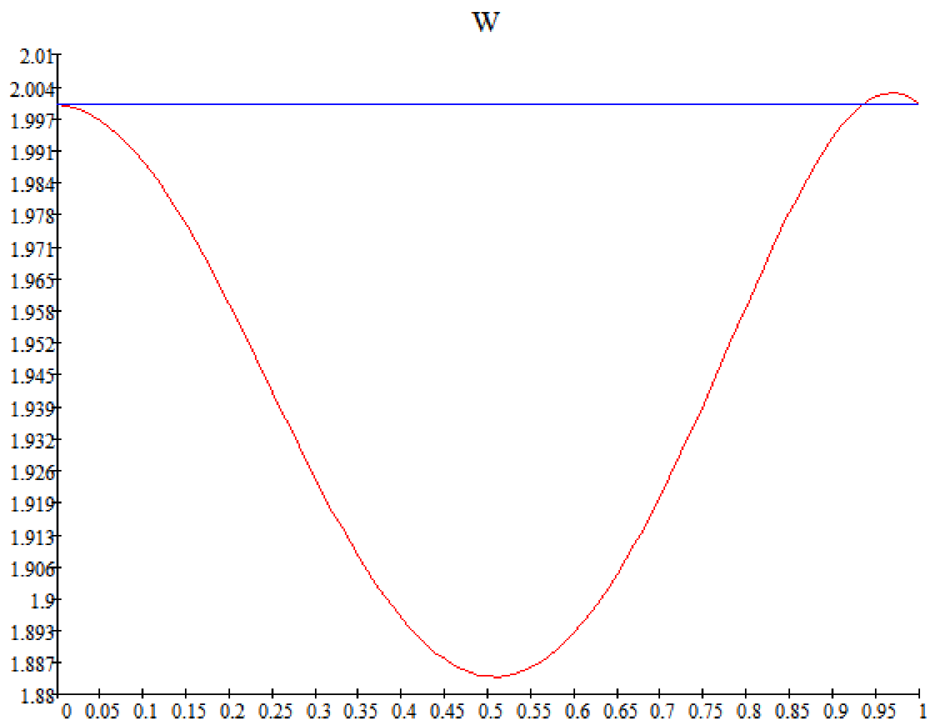

Fig. 10 ($t = 0.128$)

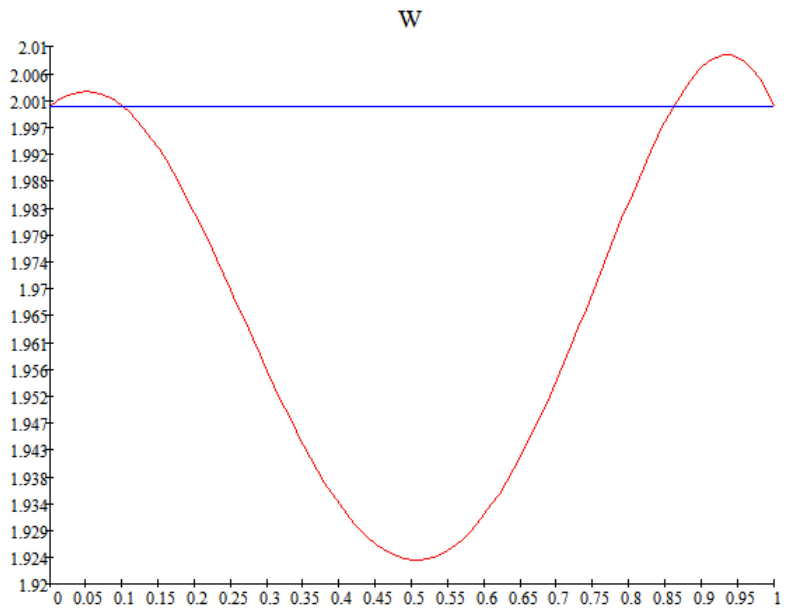

Fig. 11 ($t = 0.132$)

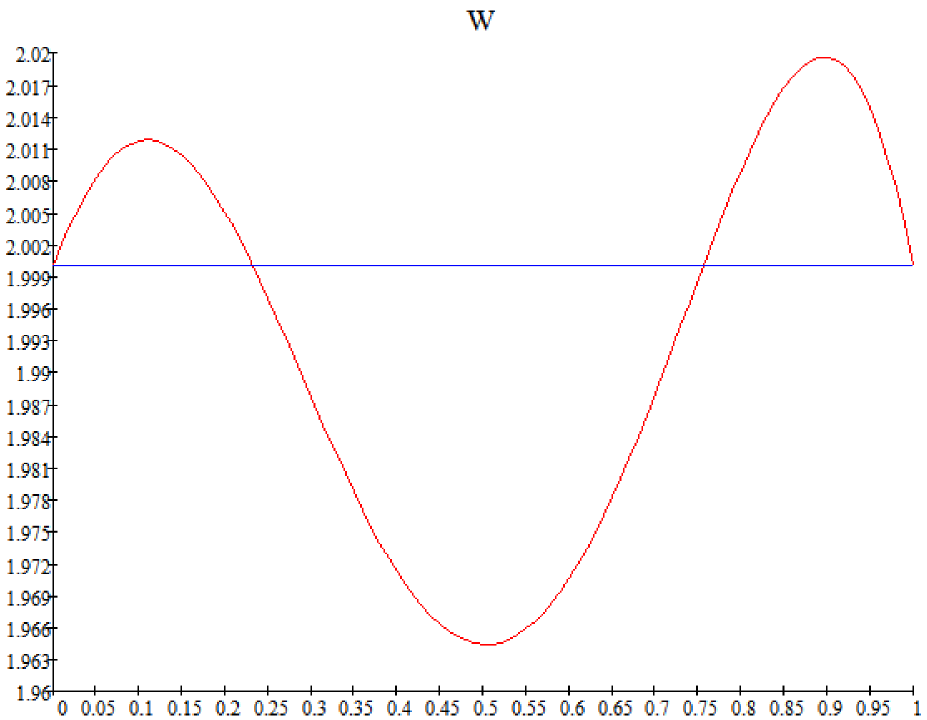

Fig. 12 ($t = 0.136$)

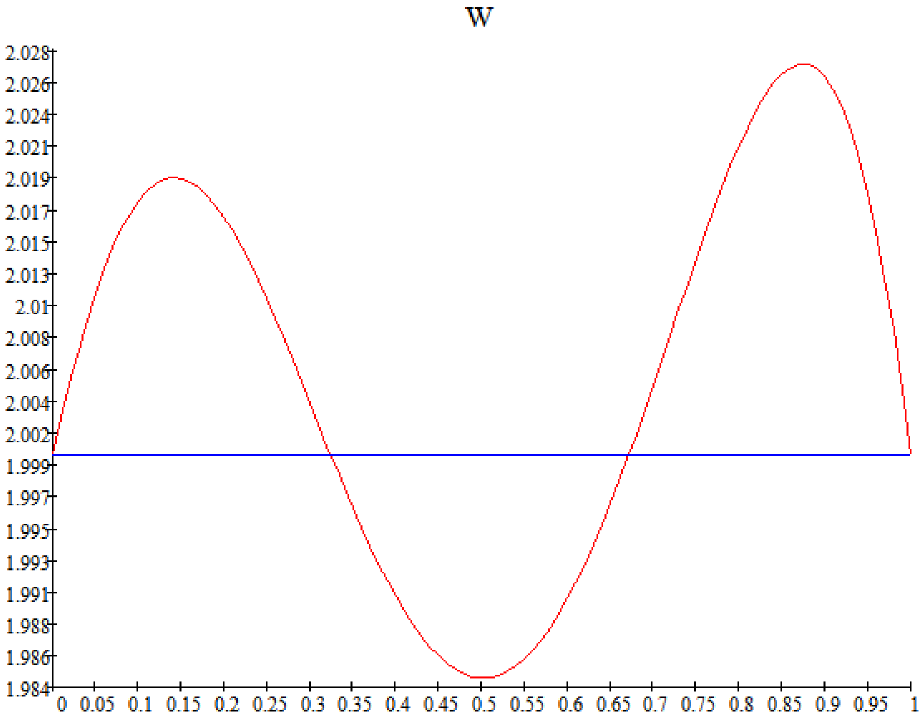

Fig. 13 ($t = 0.138$)

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წარმოადგინა ცხუმ-აფხაზეთის მეცნიერებათა აკადემიის ილია ვეკუას სახელობის მათემატიკის ინსტიტუტმა

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